Multiple critical points
for a class of nonlinear functionals [*]

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Abstract

In this paper we prove a multiplicity result concerning the critical points of a class of functionals involving local and nonlocal nonlinearities. We apply our result to the nonlinear Schrödinger-Maxwell system in $\mathbb{R}^3$ and to the nonlinear elliptic Kirchhoff equation in $\mathbb{R}^N$ assuming on the *local* nonlinearity the general hypotheses introduced by Berestycki and Lions.

1 Introduction

In the celebrated papers [8, 9], Berestycki and Lions proved the existence of a ground state and a multiplicity result for the equation

$$- \Delta u = g(u), \ u : \mathbb{R}^N \to \mathbb{R}, \tag{1}$$

for $N \geqslant 3$, assuming that

(g1) $g \in C(\mathbb{R}, \mathbb{R})$ and odd;

(g2) $-\infty < \liminf_{s \to 0^+} g(s)/s \leqslant \limsup_{s \to 0^+} g(s)/s = -m < 0$;

(g3) $-\infty \leqslant \limsup_{s \to +\infty} g(s)/s^{2^*-1} \leqslant 0$, with $2^* = 2N/(N-2)$;

(g4) there exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(s)\,ds > 0$.

Modifying, if necessary, in a suitable way the nonlinearity g (without losing the generality of the problem), it can be proved that equation (1) possesses a variational structure, namely its solutions can be found as critical points of the functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u).$$

Solutions of several nonlinear elliptic equations involving local and nonlocal nonlinearities can be found looking for critical points of a suitable perturbation of I, namely

$$I_q(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + qR(u) - \int_{\mathbb{R}^N} G(u), \ u \in H^1(\mathbb{R}^N), \tag{2}$$

where $q > 0$ is a small parameter and $R : H^1(\mathbb{R}^N) \to \mathbb{R}$. In order to define the functional I_q we need to replace (g3) with the stronger assumption

(g3)′ $\lim_{s \to +\infty} g(s)/|s|^{2^*-1} = 0$.

In this paper we are interested in providing a multiplicity result in critical point theory for I_q. To this end we suppose that $R = \sum_{i=1}^k R_i$ and, for each $i = 1, \dots, k$ the functional R_i satisfies:

(R1) R_i is $C^1(H^1(\mathbb{R}^N), \mathbb{R})$, nonnegative and even;

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(R2) there exists $\delta_i > 0$ such that $R_i'(u)[u] \leqslant C\|u\|^{\delta_i}$, for any $u \in H^1(\mathbb{R}^N)$;

(R3) if $\{u_j\}_j$ is a sequence in $H^1(\mathbb{R}^N)$ weakly convergent to $u \in H^1(\mathbb{R}^N)$, then

$$\limsup_j R_i'(u_j)[u - u_j] \leqslant 0;$$

(R4) there exist $\alpha_i, \beta_i \geqslant 0$ such that if $u \in H^1(\mathbb{R}^N)$, $t > 0$ and $u_t = u(\cdot/t)$, then

$$R_i(u_t) = t^{\alpha_i} R_i(t^{\beta_i} u);$$

(R5) R_i is invariant under the action of N-dimensional orthogonal group, i.e. $R_i(u(g\cdot)) = R_i(u(\cdot))$ for every $g \in O(N)$.

The effect deriving from the presence of the perturbation qR is to modify the structure of the functional I both as regards the geometrical properties, and as regards compactness properties. In particular two remarkable difficulties arise: the first is related with the problem of applying classical min-max arguments to find Palais-Smale sequences at suitable levels, the second is concerned with the compactness of these sequences. If, on one hand, just assuming the positiveness of the functional R we overcome the difficulty of finding suitable min-max levels, on the other, the problem of boundedness of Palais-Smale sequences is not nearly trivial. This is a consequence of the fact that no Ambrosetti-Rabinowitz hypothesis like

$$0 < \nu G(t) \leqslant tg(t), \text{ for } \nu > 2,$$

is assumed on g. The monotonicity trick based on an idea of Struwe [29] and formalized by Jeanjean [17] has turned out to be a powerful method to overcome this difficulty. By means of the monotonicity trick and a truncation argument based on an idea of Berti and Bolle [10] and of Jeanjean and Le Coz [18] (see also [21]), in [5] we have proved an existence result for a functional which is included in the class we are treating. The same arguments have been used also in [4] to prove a similar existence result also for another functional of the type described in (2). In both the results it is required that the parameter q is sufficiently small. The well known fact proved in [9] and more recently in [15] that I possesses infinitely many critical points has led us to wonder if, at least for small q, a multiplicity result on the number of critical points keeps holding for I_q. In this direction a fundamental contribution comes from the recent paper [15], where, developing some ideas of [16], a new method to find multiple solutions to equations involving general local nonlinearities has been introduced. Here we will get our multiplicity result by a suitable combination of the new method described in [15] with the truncation argument of [18].

Our main result is the following.

Theorem 1.1. *Let us suppose (g1), (g2), (g3)', (g4) and (R1)–(R5). Then for any $h \in \mathbb{N}, h \geqslant 1$, there exists $q(h) > 0$ such that for any $0 < q < q(h)$ the functional I_q admits at least h couples of critical points in $H^1(\mathbb{R}^N)$ with radial symmetry.*

Some nonlinear mathematical physics problems can be solved looking for critical points of functionals strictly related with I_q. Among them, we recall, for instance, the electrostatic Schrödinger-Maxwell equations. This system constitutes a model to describe the interaction between a nonrelativistic charged particle with the electromagnetic field (see for example [2, 3, 5, 6, 7, 11, 12, 13, 19, 20, 21, 27, 30, 32]). In the electrostatic case the system becomes

$$\begin{cases} -\Delta u + q\phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = qu^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{3}$$

Finding solutions to the previous system is equivalent to look for critical points of the functional

$$I_q(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * u^2 \right) u^2 - \int_{\mathbb{R}^3} G(u).$$

In [2], the authors study (3) with $g(u) = -u + |u|^{p-1}u$ and $1 < p < 5$ and use an abstract tool, based on the monotonicity trick, to prove a multiplicity result.

As a consequence of Theorem 1.1 we prove

Theorem 1.2. *Let us suppose (g1), (g2), (g3), (g4). Then for any $h \in \mathbb{N}, h \geqslant 1$, there exists $q(h) > 0$ such that for any $0 < q < q(h)$ system (3) admits at least h couples of solutions in $H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ with radial symmetry.*

Another variational problem related with our abstract result is the following. Let us consider the multidimensional Kirchhoff equation

$$\frac{\partial^2 u}{\partial t^2} - \left(p + q \int_\Omega |\nabla u|^2\right) \Delta u = 0 \qquad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$, $p > 0$ and u satisfies some initial or boundary conditions. It arises from the following Kirchhoff' nonlinear generalization (see [22]) of the well known d'Alembert equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0,$$

and it describes a vibrating string, taking into account the changes in length of the string during the vibration. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension.

If we look for static solutions, the equation we have to solve is

$$-\left(p + q \int_\Omega |\nabla u|^2\right) \Delta u = 0.$$

In the same spirit of [1, 4] we consider the semilinear perturbation

$$-\left(p + q \int_\Omega |\nabla u|^2\right) \Delta u = g(u), \quad \text{in } \Omega \subset \mathbb{R}^N. \tag{4}$$

Recently this equation has been extensively treated by many authors in bounded domains, assuming Dirichlet conditions on the boundary (see for example [1, 14, 23, 24, 25, 26, 31]).

Here we are interested in showing an application of our abstract result to the equation (4) in all the space $\mathbb{R}^N$, $N \geqslant 3$. The solutions are the critical points of the functional

$$I_q(u) = \frac{p}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{q}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^N} G(u).$$

We prove the following result.

Theorem 1.3. *Let us suppose (g1), (g2), (g3), (g4). Then for any $h \in \mathbb{N}, h \geqslant 1$, there exists $q(h) > 0$ such that for any $0 < q < q(h)$ equation (4) admits at least h couples of solutions in $H^1(\mathbb{R}^N)$ with radial symmetry.*

The paper is organized as follows: in Section 2 we prove Theorem 1.1; in Section 3 we show as it can be applied to the nonlinear Schrödinger-Maxwell system and the nonlinear elliptic Kirchhoff equation in order to prove Theorems 1.2 and 1.3.

NOTATION

We will use the following notations:

- for any $1 \leqslant s \leqslant +\infty$, we denote by $\|\cdot\|_s$ the usual norm of the Lebesgue space $L^s(\mathbb{R}^N)$;

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the norm
$$\|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 + u^2;$$

- $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is completion of $C_0^\infty(\mathbb{R}^N)$ (the compactly supported functions in $C^\infty(\mathbb{R}^N)$) with respect to the norm
$$\|u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\nabla u|^2;$$

- C, C', C_i are various positive constants which may also vary from line to line.

2   The abstract result

We set for any $s \geqslant 0$,

$$g_1(s) := (g(s) + ms)^+,$$
$$g_2(s) := g_1(s) - g(s),$$

and we extend them as odd functions. Since

$$\lim_{s \to 0} \frac{g_1(s)}{s} = 0,$$

$$\lim_{s \to \pm\infty} \frac{g_1(s)}{|s|^{2^*-1}} = 0, \tag{5}$$

and

$$g_2(s) \geqslant ms, \quad \forall s \geqslant 0, \tag{6}$$

by some computations, we have that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$g_1(s) \leqslant C_\varepsilon |s|^{2^*-1} + \varepsilon g_2(s), \quad \forall s \geqslant 0. \tag{7}$$

If we set

$$G_i(t) := \int_0^t g_i(s)\,ds, \quad i = 1, 2,$$

then, by (6) and (7), we have

$$G_2(s) \geqslant \frac{m}{2} s^2, \quad \forall s \in \mathbb{R} \tag{8}$$

and for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$G_1(s) \leqslant C_\varepsilon |s|^{2^*} + \varepsilon G_2(s), \quad \forall s \in \mathbb{R}. \tag{9}$$

Since, for any $u \in H^1(\mathbb{R}^N)$, $R_i(u) - R_i(0) = \int_0^1 \frac{d}{dt} R_i(tu)dt$, by (R2) we have that

$$R_i(u) \leqslant C_1 + C_2 \|u\|^{\delta_i}. \tag{10}$$

The hypothesis (R5) assures that all functionals that we will consider in this paper are invariant under rotations. Then

$$H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) \mid u \text{ radial }\}$$

is a natural constraint to look for critical points, namely critical points of the functional restricted to $H_r^1(\mathbb{R}^N)$ are *true* critical points in $H^1(\mathbb{R}^N)$. Therefore, from now on, we will directly define our functionals in $H_r^1(\mathbb{R}^N)$.

As in [18], we consider a cut-off function $\chi \in C^\infty(\mathbb{R}_+, \mathbb{R})$ such that

$$\begin{cases} \chi(s) = 1, & \text{for } s \in [0, 1], \\ 0 \leqslant \chi(s) \leqslant 1, & \text{for } s \in]1, 2[, \\ \chi(s) = 0, & \text{for } s \in [2, +\infty[, \\ \|\chi'\|_\infty \leqslant 2, \end{cases}$$

and we introduce the following truncated functional $I_q^T : H_r^1(\mathbb{R}^N) \to \mathbb{R}$

$$I_q^T(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + q k_T(u) R(u) - \int_{\mathbb{R}^3} G(u),$$

where

$$k_T(u) = \chi\left(\frac{\|u\|^2}{T^2}\right).$$

Of course, any critical point u of I_q^T with $\|u\| \leqslant T$ is a critical point of I_q.

The C^1-functional I_q^T has the symmetric mountain pass geometry:

Lemma 2.1. *There exist $r_0 > 0$ and $\rho_0 > 0$ such that*

$$I_q^T(u) \geqslant 0, \quad \text{for } \|u\| \leqslant r_0, \tag{11}$$

$$I_q^T(u) \geqslant \rho_0, \quad \text{for } \|u\| = r_0. \tag{12}$$

Moreover, for any $n \in \mathbb{N}, n \geqslant 1$, there exists an odd continuous map

$$\gamma_n : S^{n-1} = \{\sigma = (\sigma_1, \cdots, \sigma_n) \in \mathbb{R}^n \mid |\sigma| = 1\} \to H_r^1(\mathbb{R}^N),$$

such that

$$I_q^T(\gamma_n(\sigma)) < 0, \qquad \text{for all } \sigma \in S^{n-1}.$$

Proof   By (8), (9) and the positivity of the map R,

$$I_q^T(u) \geqslant C_1 \|u\|^2 - C_2 \|u\|^{2^*}$$

from which we obtain (11) and (12).

Moreover, arguing as in [9, Theorem 10], for every $n \geqslant 1$ we can consider an odd continuous map $\pi_n : S^{n-1} \to H_r^1(\mathbb{R}^N)$ such that

$$0 \notin \pi_n(S^{n-1}), \qquad \int_{\mathbb{R}^N} G(\pi_n(\sigma)) \geqslant 1 \text{ for all } \sigma \in S^{n-1}.$$

Then, for t sufficiently large, we take

$$\gamma_n(\sigma) = \pi_n(\sigma)(\cdot/t)$$

and we obtain

$$\begin{aligned}
I_q^T(\gamma_n(\sigma)) &= \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla \pi_n(\sigma)|^2 + q\chi \left(\frac{t^{N-2} \|\nabla \pi_n(\sigma)\|_2^2 + t^N \|\pi_n(\sigma)\|_2^2}{T^2} \right) R(\gamma_n(\sigma)) \\
&\quad - t^N \int_{\mathbb{R}^N} G(\pi_n(\sigma)) \\
&\leqslant \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla \pi_n(\sigma)|^2 - t^N < 0.
\end{aligned}$$

$\square$

Let us define

$$b_n = b_n(q, T) = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} I_q^T(\gamma(\sigma))$$

where $D_n = \{\sigma = (\sigma_1, \cdots, \sigma_n) \in \mathbb{R}^n \mid |\sigma| \leqslant 1\}$,

$$\Gamma_n = \left\{ \gamma \in C(D_n, H_r^1(\mathbb{R}^N)) \,\middle|\, \begin{array}{ll} \gamma(-\sigma) = -\gamma(\sigma) & \text{for all } \sigma \in D_n \\ \gamma(\sigma) = \gamma_n(\sigma) & \text{for all } \sigma \in \partial D_n \end{array} \right\}$$

and $\gamma_n : \partial D_n \to H_r^1(\mathbb{R}^N)$ is given in Lemma 2.1.

Analogously to [15], we set

$$\tilde{I}_q(\theta, u) = I_q(u(e^{-\theta} \cdot)),$$

$$\tilde{I}_q^T(\theta, u) = I_q^T(u(e^{-\theta} \cdot)),$$

$$\tilde{I}_q'(\theta, u) = \frac{\partial}{\partial u} \tilde{I}_q(\theta, u),$$

$$(\tilde{I}_q^T)'(\theta, u) = \frac{\partial}{\partial u} \tilde{I}_q^T(\theta, u),$$

$$\tilde{b}_n = \tilde{b}_n(q, T) = \inf_{\tilde{\gamma} \in \tilde{\Gamma}_n} \max_{\sigma \in D_n} \tilde{I}_q^T(\tilde{\gamma}(\sigma)),$$

where

$$\tilde{\Gamma}_n = \left\{ \tilde{\gamma} \in C(D_n, \mathbb{R} \times H_r^1(\mathbb{R}^N)) \,\middle|\, \begin{array}{ll} \tilde{\gamma}(\sigma) = (\theta(\sigma), \eta(\sigma)) \text{ satisfies} & \\ (\theta(-\sigma), \eta(-\sigma)) = (\theta(\sigma), -\eta(\sigma)) & \text{for all } \sigma \in D_n \\ (\theta(\sigma), \eta(\sigma)) = (0, \gamma_n(\sigma)) & \text{for all } \sigma \in \partial D_n \end{array} \right\}.$$

By (R4) we have

$$\tilde{I}_q(\theta, u) = \frac{e^{(N-2)\theta}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + q \sum_{i=1}^{k} e^{\alpha_i \theta} R_i(e^{\beta_i \theta} u) - e^{N\theta} \int_{\mathbb{R}^N} G(u),$$

$$\tilde{I}_q^T(\theta, u) = \frac{e^{(N-2)\theta}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + q\chi \left(\frac{e^{(N-2)\theta} \|\nabla u\|_2^2 + e^{N\theta} \|u\|_2^2}{T^2} \right) \sum_{i=1}^{k} e^{\alpha_i \theta} R_i(e^{\beta_i \theta} u) - e^{N\theta} \int_{\mathbb{R}^N} G(u).$$

Arguing as in [15], the following lemmas hold.

Lemma 2.2. *We have*

1. *there exists $\bar{b} > 0$ such that $b_n \geqslant \bar{b}$, for any $n \geqslant 1$;*

2. *$b_n \to +\infty$;*

3. *$b_n = \tilde{b}_n$, for any $n \geqslant 1$.*

Lemma 2.3. *For any $n \geqslant 1$, there exists a sequence $\{(\theta_j, u_j)\}_j \subset \mathbb{R} \times H_r^1(\mathbb{R}^N)$ such that*

(i) *$\theta_j \to 0$;*

(ii) *$\tilde{I}_q^T(\theta_j, u_j) \to b_n$;*

(iii) *$(\tilde{I}_q^T)'(\theta_j, u_j) \to 0$ strongly in $(H_r^1(\mathbb{R}^N))^{-1}$;*

(iv) *$\frac{\partial}{\partial \theta} \tilde{I}_q^T(\theta_j, u_j) \to 0$.*

Now we prove that for a suitable choice of T and q, the sequence $\{u_j\}_j$ obtained in Lemma 2.3 actually is a bounded Palais-Smale sequence fot I_q.

Proposition 2.4. *Let $n \geqslant 1$ and $T_n > 0$ sufficiently large. There exists q_n which depends on T_n, such that for any $0 < q < q_n$, if $\{(\theta_j, u_j)\}_j \subset \mathbb{R} \times H_r^1(\mathbb{R}^N)$ is the sequence given in Lemma 2.3, then, up to a subsequence, $\|u_j\| \leqslant T_n$, for any $j \geqslant 1$.*

Proof   By Lemmas 2.2 and 2.3, we infer that

$$N\tilde{I}_q^T(\theta_j, u_j) - \frac{\partial}{\partial \theta} \tilde{I}_q^T(\theta_j, u_j) = Nb_n + o_j(1),$$

and so

$$
\begin{aligned}
e^{(N-2)\theta_j} \int_{\mathbb{R}^N} |\nabla u_j|^2 = {} & q\chi \left(\frac{\|u_j(e^{-\theta_j} \cdot)\|^2}{T^2} \right) \sum_{i=1}^{k} (\alpha_i - N) R_i(u_j(e^{-\theta_j} \cdot)) \\
& + q\chi \left(\frac{\|u_j(e^{-\theta_j} \cdot)\|^2}{T^2} \right) \sum_{i=1}^{k} e^{\alpha_i \theta_j} R_i'(e^{\beta_i \theta_j} u_j)[\beta_i e^{\beta_i \theta_j} u_j] \\
& + q\chi' \left(\frac{\|u_j(e^{-\theta_j} \cdot)\|^2}{T^2} \right) \frac{(N-2)e^{(N-2)\theta_j} \|\nabla u_j\|_2^2 + Ne^{N\theta_j} \|u_j\|_2^2}{T^2} R(u_j(e^{-\theta_j} \cdot)) \\
& + Nb_n + o_j(1).
\end{aligned}
\tag{13}
$$

We are going to estimate the right part of the previous identity. By the min-max definition of b_n, if $\gamma \in \Gamma_n$, we have

$$
\begin{aligned}
b_n & \leqslant \max_{\sigma \in D_n} I_q^T(\gamma(\sigma)) \\
& \leqslant \max_{\sigma \in D_n} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \gamma(\sigma)|^2 - \int_{\mathbb{R}^N} G(\gamma(\sigma)) \right\} + \max_{\sigma \in D_n} \{qk_T(\gamma(\sigma)) R(\gamma(\sigma))\} \\
& = A_1 + A_2(T)
\end{aligned}
$$

Now, if $\|\gamma(\sigma)\|^2 \geqslant 2T^2$ then $A_2(T) = 0$, otherwise, by (10), we have

$$A_2(T) \leqslant q(C_1 + C_2\|\gamma(\sigma)\|^\delta) \leqslant q(C_1 + C_2'T^\delta),$$

for a suitable $\delta > 0$. Moreover we have that

$$q\chi\left(\frac{\|u_j(e^{-\theta_j}\cdot)\|^2}{T^2}\right)\sum_{i=1}^k(\alpha_i - N)R_i(u_j(e^{-\theta_j}\cdot)) \leqslant q(C_1 + C_2T^\delta);$$

$$q\chi\left(\frac{\|u_j(e^{-\theta_j}\cdot)\|^2}{T^2}\right)\sum_{i=1}^k e^{\alpha_i\theta_j}R_i'(e^{\beta_i\theta_j}u_j)[\beta_i e^{\beta_i\theta_j}u_j] \leqslant CqT^\delta; \tag{14}$$

$$q\chi'\left(\frac{\|u_j(e^{-\theta_j}\cdot)\|^2}{T^2}\right)\frac{(N-2)e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + Ne^{N\theta_j}\|u_j\|_2^2}{T^2}R(u_j(e^{-\theta_j}\cdot)) \leqslant q(C_1 + C_2T^\delta). \tag{15}$$

Then, from (13) we deduce that

$$\int_{\mathbb{R}^N}|\nabla u_j|^2 \leqslant C' + q(C_1 + C_2T^\delta). \tag{16}$$

On the other hand, since $\frac{\partial}{\partial\theta}\tilde{I}_q^T(\theta_j, u_j) = o_j(1)$, by (9) we have that

$$\frac{(N-2)e^{(N-2)\theta_j}}{2}\int_{\mathbb{R}^N}|\nabla u_j|^2 + q\chi\left(\frac{\|u_j(e^{-\theta_j}\cdot)\|^2}{T^2}\right)\sum_{i=1}^k\alpha_i R_i(u_j(e^{-\theta_j}\cdot))$$

$$+ q\chi\left(\frac{\|u_j(e^{-\theta_j}\cdot)\|^2}{T^2}\right)\sum_{i=1}^k e^{\alpha_i\theta_j}R_i'(e^{\beta_i\theta_j}u_j)[\beta_i e^{\beta_i\theta_j}u_j]$$

$$+ q\chi'\left(\frac{\|u_j(e^{-\theta_j}\cdot)\|^2}{T^2}\right)\frac{(N-2)e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + Ne^{N\theta_j}\|u_j\|_2^2}{T^2}R(u_j(e^{-\theta_j}\cdot))$$

$$+ Ne^{N\theta_j}\int_{\mathbb{R}^N}G_2(u_j) = Ne^{N\theta_j}\int_{\mathbb{R}^N}G_1(u_j) + o_j(1)$$

$$\leqslant Ne^{N\theta_j}\left(C_\varepsilon\int_{\mathbb{R}^N}|u_j|^{2^*} + \varepsilon\int_{\mathbb{R}^N}G_2(u_j)\right) + o_j(1). \tag{17}$$

Now, by (8), (14), (15), (16) and (17), we obtain

$$\frac{Ne^{N\theta_j}m(1-\varepsilon)}{2}\int_{\mathbb{R}^N}u_j^2 \leqslant (1-\varepsilon)Ne^{N\theta_j}\int_{\mathbb{R}^N}G_2(u_j)$$

$$\leqslant Ne^{N\theta_j}C_\varepsilon\int_{\mathbb{R}^N}|u_j|^{2^*} - q\chi\left(\frac{\|u_j(e^{-\theta_j}\cdot)\|^2}{T^2}\right)\sum_{i=1}^k e^{\alpha_i\theta_j}R_i'(e^{\beta_i\theta_j}u_j)[\beta_i e^{\beta_i\theta_j}u_j]$$

$$- q\chi'\left(\frac{\|u_j(e^{-\theta_j}\cdot)\|^2}{T^2}\right)\frac{(N-2)e^{(N-2)\theta_j}\|\nabla u_j\|_2^2 + Ne^{N\theta_j}\|u_j\|_2^2}{T^2}R(u_j(e^{-\theta_j}\cdot)) + o_j(1)$$

$$\leqslant C\left(\int_{\mathbb{R}^N}|\nabla u_j|^2\right)^{2^*/2} + q(C_1 + C_2T^\delta) + o_j(1)$$

$$\leqslant C(C' + q(C_1 + C_2T^\delta))^{2^*/2} + q(C_1 + C_2T^\delta) + o_j(1). \tag{18}$$

We suppose by contradiction that there exists no subsequence of $\{u_j\}_j$ which is uniformly bounded by T in the H^1-norm. As a consequence, for a certain j_0 it should result that

$$\|u_j\| > T, \quad \forall j \geqslant j_0. \tag{19}$$

Without any loss of generality, we are supposing that (19) is true for any u_j.
Therefore, by (16) and (18), we conclude that

$$T^2 < \|u_j\|^2 \leqslant C_3 + C_4qT^{\frac{2^*}{2}\delta}$$

which is not true for T large and q small enough: indeed we can find $T_0 > 0$ such that $T_0^2 > C_3 + 1$ and $q_0 = q_0(T_0)$ such that $C_4 q T^{\frac{2^*}{2}\delta} < 1$, for any $q < q_0$, and we find a contradiction. $\square$

In our arguments, the following variant of the Strauss' compactness result [28] (see also [8, Theorem A.1]) will be a fundamental tool.

Proposition 2.5. *Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\lim_{s \to \infty} \frac{P(s)}{Q(s)} = 0,$$

$\{v_j\}_j$, v and w be measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$, with w bounded, such that

$$\sup_j \int_{\mathbb{R}^N} |Q(v_j(x))w| \, dx < +\infty,$$

$$P(v_j(x)) \to v(x) \text{ a.e. in } \mathbb{R}^N.$$

Then $\|(P(v_j) - v)w\|_{L^1(B)} \to 0$, for any bounded Borel set B.
   Moreover, if we have also

$$\lim_{s \to 0} \frac{P(s)}{Q(s)} = 0,$$

$$\lim_{|x| \to \infty} \sup_j |v_j(x)| = 0,$$

then $\|(P(v_j) - v)w\|_{L^1(\mathbb{R}^N)} \to 0$.

In analogy with the well-known compactness result in [9], we state the following result.

Lemma 2.6. *Let $n \geqslant 1$, $T_n, q_n > 0$ as in Proposition 2.4 and $\{(\theta_j, u_j)\}_j \subset \mathbb{R} \times H_r^1(\mathbb{R}^N)$ be the sequence given in Lemma 2.3. Then $\{u_j\}_j$ admits a subsequence which converges in $H_r^1(\mathbb{R}^N)$ to a nontrivial critical point of I_q at level b_n.*

Proof   Since $\{u_j\}_j$ is bounded, up to a subsequence, we can suppose that there exists $u \in H_r^1(\mathbb{R}^N)$ such that

$$\begin{aligned}
&u_j \rightharpoonup u \ \text{ weakly in } H_r^1(\mathbb{R}^N), \\
&u_j \to u \ \text{ in } L^p(\mathbb{R}^N), \ 2 < p < 2^*, \\
&u_j \to u \ \text{ a.e. in } \mathbb{R}^N.
\end{aligned} \tag{20}$$

By weak lower semicontinuity we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 \leqslant \liminf_j \int_{\mathbb{R}^N} |\nabla u_j|^2. \tag{21}$$

Since $\|u_j\| \leqslant T_n$ we have

$$\begin{aligned}
\tilde{I}_q'(\theta_j, u_j)[v] &= (\tilde{I}_q^T)'(\theta_j, u_j)[v] \\
&= e^{(N-2)\theta_j} \int_{\mathbb{R}^N} \nabla u_j \cdot \nabla v + q \sum_{i=1}^k e^{(\alpha_i+\beta_i)\theta_j} R_i'(e^{\beta_i \theta_j} u_j)[v] \\
&\quad + e^{N\theta_j} \int_{\mathbb{R}^N} g_2(u_j)v - e^{N\theta_j} \int_{\mathbb{R}^N} g_1(u_j)v
\end{aligned}$$

for every $v \in H^1(\mathbb{R}^N)$.
Then, by (iii) of Lemma 2.3

$$\begin{aligned}
&\tilde{I}_q'(\theta_j, u_j)[u] - \tilde{I}_q'(\theta_j, u_j)[u_j] \\
&= e^{(N-2)\theta_j} \int_{\mathbb{R}^N} \nabla u_j \cdot (\nabla u - \nabla u_j) + q \sum_{i=1}^k e^{(\alpha_i+\beta_i)\theta_j} R_i'(e^{\beta_i \theta_j} u_j)[u - u_j] \\
&\quad + e^{N\theta_j} \int_{\mathbb{R}^N} g_2(u_j)(u - u_j) - e^{N\theta_j} \int_{\mathbb{R}^N} g_1(u_j)(u - u_j) = o_j(1). \tag{22}
\end{aligned}$$

If we apply Proposition 2.5 for $P(s) = g_i(s)$, $i = 1, 2$, $Q(s) = |s|^{2^*-1}$, $(v_j)_j = (u_j)_j$, $v = g_i(u)$, $i = 1, 2$ and w a generic $C_0^\infty(\mathbb{R}^N)$-function, by (g3)', (5) and (20) we deduce that

$$\int_{\mathbb{R}^N} g_i(u_j)w \to \int_{\mathbb{R}^N} g_i(u)w \quad i = 1, 2,$$

and so

$$\int_{\mathbb{R}^N} g_i(u_j)u \to \int_{\mathbb{R}^N} g_i(u)u \quad i = 1, 2. \tag{23}$$

Moreover, applying Proposition 2.5 for $P(s) = g_1(s)s$, $Q(s) = s^2 + |s|^{2^*}$, $(v_j)_j = (u_j)_j$, $v = g_1(u)u$, and $w = 1$, by (g3)', (5) and (20), we deduce that

$$\int_{\mathbb{R}^N} g_1(u_j)u_j \to \int_{\mathbb{R}^N} g_1(u)u. \tag{24}$$

Moreover, by (20) and Fatou's lemma

$$\int_{\mathbb{R}^N} g_2(u)u \leqslant \liminf_j \int_{\mathbb{R}^N} g_2(u_j)u_j. \tag{25}$$

By (22), (23), (24) (25) and (R3), we have

$$\limsup_j \int_{\mathbb{R}^N} |\nabla u_j|^2 = \limsup_j e^{(N-2)\theta_j} \int_{\mathbb{R}^N} |\nabla u_j|^2$$

$$= \limsup_j \left[e^{(N-2)\theta_j} \int_{\mathbb{R}^N} \nabla u_j \cdot \nabla u + q \sum_{i=1}^k e^{(\alpha_i + \beta_i)\theta_j} R_i'(e^{\beta_i \theta_j} u_j)[u - u_j] \right.$$

$$\left. + e^{N\theta_j} \int_{\mathbb{R}^N} g_2(u_j)(u - u_j) - e^{N\theta_j} \int_{\mathbb{R}^N} g_1(u_j)(u - u_j) \right]$$

$$\leqslant \int_{\mathbb{R}^N} |\nabla u|^2. \tag{26}$$

By (21) and (26), we get

$$\lim_j \int_{\mathbb{R}^N} |\nabla u_j|^2 = \int_{\mathbb{R}^N} |\nabla u|^2, \tag{27}$$

hence, by (22),

$$\lim_j \int_{\mathbb{R}^N} g_2(u_j)u_j = \int_{\mathbb{R}^N} g_2(u)u. \tag{28}$$

Since $g_2(s)s = ms^2 + h(s)$, with h a positive and continuous function, by Fatou's Lemma we have

$$\int_{\mathbb{R}^N} h(u) \leqslant \liminf_j \int_{\mathbb{R}^N} h(u_j),$$

$$\int_{\mathbb{R}^N} u^2 \leqslant \liminf_j \int_{\mathbb{R}^N} u_j^2.$$

These last two inequalities and (28) imply that, up to a subsequence,

$$\lim_j \int_{\mathbb{R}^N} u_j^2 = \int_{\mathbb{R}^N} u^2,$$

which, together with (27), shows that $u_j \to u$ strongly in $H_r^1(\mathbb{R}^N)$. Therefore, since $b_n > 0$, u is a non-trivial critical point of I_q at level b_n. $\quad\square$

Proof of Theorem 1.1 Let $h \geqslant 1$. Since $b_n \to +\infty$, up to a subsequence, we can consider $b_1 < b_2 < \cdots < b_h$. By Lemma 2.6 we conclude, defining $q(h) = q_h > 0$. $\quad\square$

3   Some applications

3.1   The nonlinear Schrödinger-Maxwell system

Let us consider the Schrödinger-Maxwell system:

$$\begin{cases} -\Delta u + q\phi u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = qu^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{SM}$}$$

where $q > 0$ and g satisfies (g1)-(g4). Arguing as in [5, 8], without loss of generality, we can suppose that g satisfies (g3)'.

The solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ of ($\mathcal{SM}$) are the critical points of the action functional $\mathcal{E}_q \colon H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$\mathcal{E}_q(u, \phi) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 + \frac{q}{2} \int_{\mathbb{R}^3} \phi u^2 - \int_{\mathbb{R}^3} G(u).$$

The action functional $\mathcal{E}_q$ exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinite dimensional subspaces. This indefiniteness can be removed using the reduction method described in [7], by which we are led to study a one variable functional that does not present such a strongly indefinite nature. Indeed, for every $u \in L^{\frac{12}{5}}(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of

$$-\Delta \phi = qu^2, \qquad \text{in } \mathbb{R}^3.$$

Moreover it can be proved that $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of ($\mathcal{SM}$) (critical point of functional $\mathcal{E}_q$) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional $I_q \colon H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$I_q(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} G(u),$$

and $\phi = \phi_u$.

According to our notations, in this case $R(u) = \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2$. In order to check that R satisfies (R1)-(R5), we need some preliminary results on ϕ_u (see for example [12]).

Lemma 3.1. *The map $u \in L^{\frac{12}{5}}(\mathbb{R}^3) \mapsto \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ is C^1. Moreover, for every $u \in H^1(\mathbb{R}^3)$, we have*

  i) $\|\phi_u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = q \int_{\mathbb{R}^3} \phi_u u^2$;

  ii) $\phi_u \geqslant 0$;

  iii) $\phi_{-u} = \phi_u$;

  iv) *for any $t > 0$: $\phi_{u_t}(x) = t^2 \phi_u(x/t)$, where $u_t(x) = u(x/t)$;*

  v) *there exist $C, C' > 0$ independent of $u \in H^1(\mathbb{R}^3)$ such that*

$$\|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leqslant Cq\|u\|^2_{\frac{12}{5}},$$

  and

$$\int_{\mathbb{R}^3} \phi_u u^2 \leqslant C'q\|u\|^4_{\frac{12}{5}}; \tag{29}$$

  vi) *if u is a radial function then ϕ_u is radial, too.*

Now we use the previous lemma to deduce assumptions (R1)-(R5).
Hypothesis (R1) is obvious.
Since

$$R'(u)[u] = \int_{\mathbb{R}^3} \phi_u u^2,$$

(see for example [7]), then (R2) is again a consequence of (29).
We pass to check (R3). Suppose that

$$u_j \rightharpoonup u \text{ weakly in } H^1_r(\mathbb{R}^3).$$

By compact embedding we deduce that

$$u_j \to u \text{ in } L^{\frac{12}{5}}(\mathbb{R}^3)$$

and then, by continuity,

$$\phi_{u_j} \to \phi_u \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^3).$$

Since $R'(u)[v] = \int_{\mathbb{R}^N} \phi_u uv$, we have that

$$\limsup_j R'(u_j)[u - u_j] = \limsup_j \int_{\mathbb{R}^3} \phi_{u_j} u_j (u - u_j)$$
$$\leqslant C \limsup_j \|\phi_{u_j}\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \|u_j\|_{\frac{12}{5}} \|u - u_j\|_{\frac{12}{5}} = 0.$$

Now in order to verify (R4), we consider $u \in H^1(\mathbb{R}^3)$, $u \neq 0$ and the rescaled function u_t. We compute

$$R(u_t) = \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_t} u_t^2 = \frac{t^5}{4} \int_{\mathbb{R}^3} \phi_u u^2 = t^5 R(u)$$

so (R4) holds true for $\alpha = 5$.
Finally (R5) follows from vi) of Lemma 3.1.

3.2 The elliptic Kirchhoff equation

In this subsection we treat the semilinear perturbation of the Kirchhoff equation

$$- \left(p + q \int_{\mathbb{R}^N} |\nabla u|^2 \right) \Delta u = g(u) \qquad \text{in } \mathbb{R}^N, \tag{$\mathcal{K}$}$$

where $p > 0$ and g satisfies (g1)-(g4). Arguing as in [4, 8], without loss of generality, we can suppose that g satisfies (g3)'. We find the solution to ($\mathcal{K}$) as the critical points of the functional

$$I_q(u) = \frac{1}{2} \left(p + \frac{q}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \right) \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u).$$

It is easy to see that I_q is of the type (2), where $R(u) = \frac{1}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2$.
Assumptions (R1)-(R2) are trivially satisfied as we can see by straight computations.
As to (R3), suppose that $u_j \rightharpoonup u$ weakly in $H_r^1(\mathbb{R}^N)$. By weak lower semicontinuity, we know that

$$\int_{\mathbb{R}^N} |\nabla u|^2 \leqslant \liminf_j \int_{\mathbb{R}^N} |\nabla u_j|^2,$$

and then

$$\limsup_j R'(u_j)[u - u_j] = \limsup_j \int_{\mathbb{R}^N} |\nabla u_j|^2 \cdot \int_{\mathbb{R}^N} \nabla u_j \cdot \nabla(u - u_j)$$
$$\leqslant \limsup_j \int_{\mathbb{R}^N} |\nabla u_j|^2 \cdot \limsup_j \int_{\mathbb{R}^N} \nabla u_j \cdot \nabla(u - u_j)$$
$$\leqslant \limsup_j \int_{\mathbb{R}^N} |\nabla u_j|^2 \cdot \left(\limsup_j \int_{\mathbb{R}^N} \nabla u_j \cdot \nabla u - \liminf_j \int_{\mathbb{R}^N} |\nabla u_j|^2 \right)$$
$$\leqslant \limsup_j \int_{\mathbb{R}^N} |\nabla u_j|^2 \cdot \left(\int_{\mathbb{R}^N} |\nabla u|^2 - \liminf_j \int_{\mathbb{R}^N} |\nabla u_j|^2 \right) \leqslant 0.$$

By a simple computation, we have that

$$R(u_t) = \frac{1}{4} \left(\int_{\mathbb{R}^N} |\nabla u_t|^2 \right)^2 = \frac{t^{2(N-2)}}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 = t^{2(N-2)} R(u),$$

and then also (R4) is satisfied.
Finally by a simple change of variable it can be proved that for any $g \in O(N)$ we have

$$R(u(gx)) = \frac{1}{4} \left(\int_{\mathbb{R}^N} |\nabla u(gx)|^2 \right)^2 = \frac{1}{4} \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 \right)^2 = R(u).$$

Remark 3.2. *Let us observe that we can easily apply Theorem 1.1 also to a sort of linear combination of the Schrödinger-Maxwell equation with the Kirchhoff one, namely we can find multiple critical points of the functional*

$$I_q(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{q}{4} \left[\lambda_1 \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * u^2 \right) u^2 + \lambda_2 \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 \right] - \int_{\mathbb{R}^3} G(u),$$

with $\lambda_1, \lambda_2 \in \mathbb{R}_+$ and q sufficiently small.

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